Ternary code VOA and an automorphism of order 3(Algebraic combinatorics and the related areas of research)

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Ternary code VOA and an automorphism of order 3

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1. Introduction

Let $V = (V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (VOA) and g be its automorphism of finite order. Then the space $V^g = \{v \in V \mid gv = v\}$ of fixed points of g in V is a subalgebra of V, which is called an orbifold of the VOA V. In the case where V is the lattice VOA V_L associated with a positive definite even lattice L and the automorphism g is a lift of the -1 isometry of the lattice L, the orbifold $V^g = V_L^+$ has been studied extensively. In fact, the construction of the Moonshine module $V^\natural$ by Frenkel, Lepowsky and Meurman [7] is based on the study of V_Λ^+, where Λ denotes the Leech lattice. However, it is difficult to develop the representation theory of an orbifold in general, even if the representation theory of the original VOA V is well understood. Here the representation theory of a VOA means the study of basic properties such as the rationality and the C_2-cofiniteness, etc. together with the classification of simples modules and the determination of fusion rules.

For a V-module (M, Y_M), we can define a new V-module $(M \circ g, Y_{M\circ g})$ by $M = M \circ g$ as a vector space and $Y_{M\circ g}(v, z) = Y_M(gv, z)$ for $v \in V$. Denote by $\mathcal{M}$ a complete set of representatives of isomorphism classes of simple V-modules. Then $M \mapsto M \circ g$ induces a permutation on $\mathcal{M}$. If $M \cong M \circ g$, M is said to be g-stable.

For simplicity, assume that V is rational, C_2-cofinite, and of CFT-type. There are known examples of simple V^g-modules.

(1) If $M \in \mathcal{M}$ is g-stable, then $M(\varepsilon) = \{u \in M \mid gu = \xi^\varepsilon u\}$, $0 \le \varepsilon \le |g| - 1$, are simple V^g-modules, where $\xi = \exp(2\pi\sqrt{-1}/|g|)$.

(2) If $\{M^0, M^1, \dots, M^{|g|-1}\}$ is a g-orbit in $\mathcal{M}$, then M^i, $0 \le i \le |g| - 1$, are equivalent simple V^g-modules.

(3) If $V^T(g^i)$ is a simple g^i-twisted V-module, then $V^T(g^i)(\varepsilon) = \{u \in V^T(g^i) \mid g^i u = \xi^\varepsilon u\}$, $0 \le \varepsilon \le |g| - 1$, $1 \le i \le |g| - 1$ are simple V^g-modules.

Furthermore, those simple V^g-modules are inequivalent (cf. [6, 15]). It is also known that the number of inequivalent simple g^i-twisted V-modules is less than or equal to the number of inequivalent g-stable simple V-modules (cf. [4]).

It is natural to expect that any simple V^g-module is one of the above mentioned simple V^g-modules. In fact, no simple V^g-module of other type is known so far.

In this note we shall discuss an orbifold of a certain lattice VOA related to a ternary code by an automorphism of order 3. We want to classify the simple modules for the orbifold. Although the work is not finished yet, we shall show here the first step toward the classification of simple modules.

2. VOA $(V_{\sqrt{2}A_2})^\tau$

In this section we briefly review the VOA $(V_{\sqrt{2}A_2})^\tau$, which is the fixed point subalgebra of a lattice VOA $V_{\sqrt{2}A_2}$ by a certain automorphism τ of order 3. Detailed description of $(V_{\sqrt{2}A_2})^\tau$ can be found in [17].

Let α_1, α_2 be the simple roots of type A_2 and set $\alpha_0 = -(\alpha_1+\alpha_2)$, so that $\langle \alpha_i, \alpha_i\rangle = 2$ and $\langle \alpha_i, \alpha_j\rangle = -1$ if $i \neq j$. Set $\beta_i = \sqrt{2}\alpha_i$ and $L = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2 \cong \sqrt{2}A_2$. Since $\langle \alpha, \beta\rangle \in 2\mathbb{Z}$ for all $\alpha, \beta \in L$, the central extension $\hat{L}$ of L considered in [3, 7] splits and the twisted group algebra $\mathbb{C}\{L\}$ can be identified with the ordinary group algebra $\mathbb{C}[L]$. We use the same notation as in [8] to denote cosets of L in its dual lattice $L^\perp = \{\alpha \in \mathbb{Q}\otimes_\mathbb{Z} L \mid \langle \alpha, L\rangle \subset \mathbb{Z}\}$. Thus

$$L^0 = L, \quad L^1 = \frac{-\beta_1+\beta_2}{3} + L, \quad L^2 = \frac{\beta_1 - \beta_2}{3} + L,$$
$$L_0 = L, \quad L_a = \frac{\beta_2}{2} + L, \quad L_b = \frac{\beta_0}{2} + L, \quad L_c = \frac{\beta_1}{2} + L,$$
$$L^{(i,j)} = L_i + L^j; \quad i \in \mathcal{K}, j \in \{0,1,2\},$$

where $\mathcal{K} = \{0, a, b, c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is a Klein's four-group. Note that $L^{(i,j)}, i \in \mathcal{K}, j \in \{0,1,2\}$ are all the cosets of L in $L^\perp$ and $L^\perp/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

Now, set

$$x(\alpha) = e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}, \qquad y(\alpha) = e^{\sqrt{2}\alpha} - e^{-\sqrt{2}\alpha}, \qquad w(\alpha) = \frac{1}{2}\alpha(-1)^2 - x(\alpha)$$

for $\alpha \in \{\pm\alpha_0, \pm\alpha_1, \pm\alpha_2\}$. Moreover, set

$$\omega = \frac{1}{6}\big(\alpha_1(-1)^2 + \alpha_2(-1)^2 + \alpha_0(-1)^2\big),$$
$$\tilde{\omega}^1 = \frac{1}{5}\big(w(\alpha_1) + w(\alpha_2) + w(\alpha_0)\big), \qquad \tilde{\omega}^2 = \omega - \tilde{\omega}^1,$$
$$\omega^1 = \frac{1}{4} w(\alpha_1), \qquad \omega^2 = \tilde{\omega}^1 - \omega^1.$$

Then ω is the Virasoro element of V_L and ω^1, ω^2, and $\tilde{\omega}^2$ are mutually orthogonal conformal vectors of central charge $1/2$, $7/10$, and $4/5$, respectively (cf.[5]). Note that $\tilde{\omega}^1 = \omega^1 + \omega^2$ is a conformal vector of central charge $1/2 + 7/10 = 6/5$.

The subalgebra $\mathrm{Vir}(\tilde{\omega}^i)$ generated by $\tilde{\omega}^i$ is isomorphic to a Virasoro VOA of given central charge. The commutants of $\mathrm{Vir}(\tilde{\omega}^1)$ and $\mathrm{Vir}(\tilde{\omega}^2)$ in V_L, namely,

$$M_k^0 = \{v \in V_L \mid (\tilde{\omega}^2)_1 v = 0\}, \qquad M_t^0 = \{v \in V_L \mid (\tilde{\omega}^1)_1 v = 0\}$$

are important for our discussion. They are in fact simple subalgebras of V_L. We set

$$W_k^0 = \{v \in V_L \mid (\tilde{\omega}^2)_1 v = \frac{2}{5} v\}, \qquad W_t^0 = \{v \in V_L \mid (\omega^1)_1 v = 0,\ (\omega^2)_1 v = \frac{3}{5} v\}.$$

Then W_k^0 is a simple M_k^0-module and W_t^0 is a simple M_t^0-module. We have

$$M_k^0 \cong \Big(L(\frac{1}{2},0) \otimes L(\frac{7}{10},0)\Big) \oplus \Big(L(\frac{1}{2},\frac{1}{2}) \otimes L(\frac{7}{10},\frac{3}{2})\Big), \qquad M_t^0 \cong L(\frac{4}{5},0) \oplus L(\frac{4}{5},3).$$

Furthermore, $V_L \cong (M_k^0 \otimes M_t^0) \oplus (W_k^0 \otimes W_t^0)$ as an $M_k^0 \otimes M_t^0$-module.

We consider the following two isometries of the lattice $(L, \langle\cdot,\cdot\rangle)$:

$$\theta : \beta_i \to -\beta_i, \quad i = 1, 2,$$
$$\tau : \beta_1 \to \beta_2 \to \beta_0 \to \beta_1.$$

These two isometries lift to automorphisms of the VOA V_L. We use the same symbols θ and τ to denote these automorphisms. Since the conformal vectors $\tilde{\omega}^i$, $i = 1, 2$ are fixed by θ and τ, M_k^0, M_t^0, and W_k^0 are invariant under them. However, W_t^0 is not invariant under τ, since ω^1 is not fixed by τ.

Let us introduce two important weight 3 vectors J and K.

$$\begin{aligned} J = &-\frac{1}{6}\Big(\beta_1(-2)(\beta_2-\beta_0)(-1)+\beta_2(-2)(\beta_0-\beta_1)(-1)+\beta_0(-2)(\beta_1-\beta_2)(-1)\Big)\\ &-(\beta_2-\beta_0)(-1)y(\alpha_1)-(\beta_0-\beta_1)(-1)y(\alpha_2)-(\beta_1-\beta_2)(-1)y(\alpha_0),\\ K = &-\frac{1}{9}(\beta_1-\beta_2)(-1)(\beta_2-\beta_0)(-1)(\beta_0-\beta_1)(-1)\\ &+(\beta_2-\beta_0)(-1)x(\alpha_1)+(\beta_0-\beta_1)(-1)x(\alpha_2)+(\beta_1-\beta_2)(-1)x(\alpha_0). \end{aligned}$$

Let $M(0) = (M_k^0)^\tau = \{u \in M_k^0 \mid \tau u = u\}$. The VOA $M(0)$ was studied in [2]. Among other things, the classification of simple modules, the rationality and the C_2-cofiniteness for $M(0)$ were established. It is known that $M(0)$ is a W_3 algebra of central charge 6/5 with the Virasoro element $\tilde{\omega}^1$. In fact, $M(0)$ is generated by $\tilde{\omega}^1$ and J. The VOA M_t^0 was studied in [10, 14]. It is also rational and C_2-cofinite. The classification of simple M_t^0-modules was also established. Moreover, M_t^0 is a W_3 algebra of central charge 4/5 with the Virasoro element $\tilde{\omega}^2$ and it is generated by $\tilde{\omega}^2$ and K. Since $\tilde{\omega}^2$ and K are fixed by τ, M_t^0 is contained in $V_L^\tau = \{v \in V_L \mid \tau v = v\}$.

Let $M^0 = M(0) \otimes M_t^0$, which is a subalgebra of V_L^τ. Let $W(0) = (W_k^0)^\tau = \{u \in W_k^0 \mid \tau u = u\}$ and $W^0 = W(0) \otimes W_t^0$. Then W^0 is a simple M^0-module and $V_L^\tau = M^0 \oplus W^0$. More precisely, W^0 is a simple highest weight M^0-module with highest weight vector

$$P = y(\alpha_1) + y(\alpha_2) + y(\alpha_0).$$

Actually, we have $(\tilde{\omega}^1)_n P = (\tilde{\omega}^2)_n P = 0$ for $n \geq 2$, $(\tilde{\omega}^1)_1 P = (8/5)P$, $(\tilde{\omega}^2)_1 P = (2/5)P$, and $J_n P = K_n P = 0$ for $n \geq 2$.

The VOA V_L^τ is generated by the five vectors $\tilde{\omega}^1$, $\tilde{\omega}^2$, J, K, and P. The Griess algebra of V_L^τ, that is, the weight 2 subspace is of dimension 3 and we can take $\{\tilde{\omega}^1, \tilde{\omega}^2, P\}$ as its basis.

It is known (cf. [17]) that V_L^τ is rational, C_2-cofinite, and there are exactly 30 inequivalent simple V_L^τ-modules, which are

(1) $V_{L(0,j)}(\varepsilon)$, j, $\varepsilon = 0, 1, 2$,

(2) $V_{L(c,j)}$, $j = 0, 1, 2$,

(3) $V_L^{T,j}(\tau^i)(\varepsilon)$, j, $\varepsilon = 0, 1, 2$, $i = 1, 2$.

Here $V_L^{T,j}(\tau^i)$ is a simple τ^i-twisted V_L-module. Note that $\{V_{L(x,j)} \mid x \in \mathcal{K}, j \in \{0, 1, 2\}\}$ is a complete set of representatives of isomorphism classes of simple V_L-modules by [1]. We have

$$V_{L(x,j)} \circ \tau = V_{L(\tau^{-1}(x),j)},$$

where the action of τ on $\mathcal{K}$ is defined by $\tau(0)=0$ and $\tau : a \mapsto b \mapsto c \mapsto a$. Thus $V_{L(x,j)}$ is τ-stable if and only if $x=0$. In particular, there are exactly three τ-stable simple V_L-modules. Furthermore, $\{V_{L(a,j)},\ V_{L(b,j)},\ V_{L(c,j)}\}$ is a τ-orbit under the action $M \mapsto M \circ \tau$ of τ. Thus $V_{L(a,j)}$, $V_{L(b,j)}$, and $V_{L(c,j)}$ are equivalent simple V_L^{τ}-modules.

In this case the number of inequivalent simple τ^i-twisted V_L-modules is equal to the number of inequivalent τ-stable simple V_L-modules and all the simple τ^i-twisted V_L-module $V_L^{T,j}(\tau^i)$, $j \in \{0,1,2\}$ can be obtained by the construction of Dong and Lepowsky [3, 13] (cf. [12]).

3. Lattice $L_{C\times D}$

We follow the notation in [8, 9, 10, 11]. A $\mathbb{Z}_2 \times \mathbb{Z}_2$-code of length ℓ means an additive subgroup of $\mathcal{K}^\ell$. For $x, y \in \mathcal{K}$, define

$$x \circ y = \begin{cases} 1 & \text{if } x = y \neq 0, \\ -\frac{1}{2} & \text{if } x \neq y, x \neq 0, y \neq 0, \\ 0 & \text{if } x = 0 \text{ or } y = 0, \end{cases}$$

$$x \cdot y = \begin{cases} 1 & \text{if } x \neq y, x \neq 0, y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $2(x \circ y) \equiv x \cdot y \pmod{\mathbb{Z}}$. For $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\mu = (\mu_1, \dots, \mu_\ell) \in \mathcal{K}^\ell$, let $\lambda \cdot \mu = \sum_{i=1}^{\ell} \lambda_i \cdot \mu_i \in GF(2)$. The orthogonal form $\mathcal{K}^\ell \times \mathcal{K}^\ell \to GF(2); (\lambda, \mu) \mapsto \lambda \cdot \mu$ on $\mathcal{K}^\ell$ was used in [8, 11]. For a $\mathbb{Z}_2 \times \mathbb{Z}_2$-code C of length ℓ, we define its dual code by

$$C^\perp = \{\lambda \in \mathcal{K}^\ell \mid \lambda \cdot \mu = 0 \text{ for all } \mu \in C\}.$$

A $\mathbb{Z}_2 \times \mathbb{Z}_2$-code C is said to be self-orthogonal if $C \subset C^\perp$ and self-dual if $C = C^\perp$. For $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathcal{K}^\ell$, its support is defined by $\operatorname{supp}(\lambda) = \{i \mid \lambda_i \neq 0\}$. The cardinality of $\operatorname{supp}(\lambda)$ is called the weight of λ. We denote the weight of λ by $\operatorname{wt}(\lambda)$. A $\mathbb{Z}_2 \times \mathbb{Z}_2$-code C is said to be even if $\operatorname{wt}(\lambda)$ is even for every $\lambda \in C$.

In Section 2 we consider the action of τ on $\mathcal{K}$. Note that τ also acts on $\mathcal{K}^\ell$ by $\tau(\lambda) = (\tau(\lambda_1), \dots, \tau(\lambda_\ell))$.

The following lemma can be obtained by a simple argument (cf. [8, 9]).

Lemma 3.1. *Let C be a $\mathbb{Z}_2 \times \mathbb{Z}_2$-code of length ℓ.*

(1) *If C is even, then C is self-orthogonal.*

(2) *If C is τ-invariant, then C is even if and only if C is self-orthogonal.*

We can identify $\mathbb{Z}_2 \times \mathbb{Z}_2$ with a quadratic extension $GF(4)$ of $GF(2)$ by

$$0 \leftrightarrow 0, \quad a \leftrightarrow 1, \quad b = \tau(a) \leftrightarrow \tau, \quad c = \tau(a) \leftrightarrow \tau^2.$$

If C is a τ-invariant $\mathbb{Z}_2 \times \mathbb{Z}_2$-code, then it can be regarded as a $GF(4)$-code of the same length through the above identification. Moreover, we can introduce a hermitian form $h(\lambda, \mu) = \sum_{i=1}^{\ell} \lambda_i \overline{\mu_i}$, where $\overline{0} = 0$, $\overline{1} = 1$, $\overline{\tau} = \tau^2$, and $\overline{\tau^2} = \tau$ (cf. [9]).

Lemma 3.2. ([9, Lemma 3.2]) *A τ-invariant $\mathbb{Z}_2 \times \mathbb{Z}_2$-code C is self-orthogonal if and only if it is a self-orthogonal $GF(4)$-code with respect to the hermitian form $h(\cdot, \cdot)$.*

A ternary code of length ℓ is a subspace of the vector space $GF(3)^\ell$. For $\gamma = (\gamma_1, \ldots, \gamma_\ell)$, $\delta = (\delta_1, \ldots, \delta_\ell) \in GF(3)^\ell$, we consider the ordinary inner product $\gamma \cdot \delta = \sum_{i=1}^{\ell} \gamma_i \delta_i \in GF(3)$. The dual code $D^\perp$ of a ternary code D is defined by

$$D^\perp = \{\gamma \in GF(3)^\ell \,|\, \gamma \cdot \delta = 0 \text{ for all } \delta \in D\}.$$

We define the support and the weight of $\gamma = (\gamma_1, \ldots, \gamma_\ell) \in GF(3)^\ell$ in the same way as before. That is, $\mathrm{supp}(\gamma) = \{i \,|\, \gamma_i \neq 0\}$ and $\mathrm{wt}(\gamma)$ is the cardinality of $\mathrm{supp}(\gamma)$.

Recall the 12 cosets $L^{(x,i)}$, $x \in \mathcal{K}$, $i \in GF(3)$ of $L = \sqrt{2}A_2$ in its dual lattice $L^\perp$ considered in Section 2. For each $x \in \mathcal{K}$ we assign $\beta(x) \in L^\perp$ as follows: $\beta(0) = 0$, $\beta(a) = \beta_2/2$, $\beta(b) = \beta_0/2$, and $\beta(c) = \beta_1/2$. Then

$$L^{(x,i)} = \{\beta(x) + (-\frac{i}{3} + m_1)\beta_1 + (\frac{i}{3} + m_2)\beta_2 \,|\, m_1, m_2 \in \mathbb{Z}\}.$$

We can identify $\mathcal{K}$ with $GF(2)^2$ by

$$0 \leftrightarrow (0,0), \quad a \leftrightarrow (0,1), \quad b \leftrightarrow (1,1), \quad c \leftrightarrow (1,0).$$

We also write $x \in \mathcal{K}$ as $x = (x_1, x_2) \in GF(2)^2$ by the identification. Then for

$$\begin{aligned} \alpha &= \beta(x) + (-\frac{i}{3} + m_1)\beta_1 + (\frac{i}{3} + m_2)\beta_2, \\ \beta &= \beta(y) + (-\frac{j}{3} + n_1)\beta_1 + (\frac{j}{3} + n_2)\beta_2 \end{aligned}$$

with $x, y \in \mathcal{K}$ and $i, j \in \{0, 1, 2\}$, we have

$$\begin{aligned} \langle \alpha, \beta \rangle \equiv\ & x \circ y + \frac{4}{3} i \cdot j \\ & + x_1(j + n_2) + x_2(j + n_1) + y_1(i + m_2) + y_2(i + m_1) \pmod{2\mathbb{Z}} \end{aligned} \tag{3.1}$$

and in particular,

$$\langle \alpha, \beta \rangle \equiv \frac{1}{2} x \cdot y + \frac{1}{3} i \cdot j \pmod{\mathbb{Z}}. \tag{3.2}$$

For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{K}^\ell$ and $\gamma = (\gamma_1, \ldots, \gamma_\ell) \in GF(3)^\ell$, let

$$L_{(\lambda,\gamma)} = L^{(\lambda_1,\gamma_1)} \oplus \cdots \oplus L^{(\lambda_\ell,\gamma_\ell)} \subset (L^\perp)^{\oplus \ell},$$

where $(L^\perp)^{\oplus \ell}$ is an orthogonal sum of ℓ copies of $L^\perp$. Moreover, for a $\mathbb{Z}_2 \times \mathbb{Z}_2$-code C of length ℓ and a ternary code D of the same length, set

$$L_{C \times D} = \bigcup_{\lambda \in C, \gamma \in D} L_{(\lambda,\gamma)}.$$

Then $L_{C\times D}$ is an additive subgroup of $(L^\perp)^{\oplus \ell}$. However, $L_{C\times D}$ is not an integral lattice in general. The following lemma is a direct consequence of (3.2).

Lemma 3.3. $\{\alpha \in (\mathbb{Q} \otimes_{\mathbb{Z}} L)^{\oplus \ell} \,|\, \langle \alpha, L_{C\times D} \rangle \subset \mathbb{Z}\} = L_{C^\perp \times D^\perp}$.

Thus $L_{C\times D}$ is an integral lattice if and only if both of C and D are self-orthogonal. By (3.1) and (3.2) we also have the following lemma.

Lemma 3.4. (1) *If C is even and D is self-orthogonal, then $L_{C\times D}$ is an even lattice.*
(2) *If both of C and D are self-dual, then $L_{C\times D}$ is a unimodular lattice.*

Example. Let $\ell = 4$ and

$$C = \begin{pmatrix} a & a & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & a & a \\ 0 & 0 & b & b \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix},$$

where we denote C and D by their generating matrices. Then $L_{C\times D} \cong E_8$. Note that D is a ternary tetra code. If C is the zero code, then $L_D = L_{0\times D} \cong \sqrt{2}E_8$.

In the case where $\ell = 12$ and D is a orthogonal sum of three copies of a ternary tetra code, we can choose a τ-invariant $\mathbb{Z}_2 \times \mathbb{Z}_2$-code C such that $L_{C\times D} \cong \Lambda$; the Leech lattice (cf. [8, 9]).

4. Simple modules for $(V_{L_{C\times D}})^\tau$

From now on we assume that C is a τ-invariant even $\mathbb{Z}_2 \times \mathbb{Z}_2$-code of length $\ell \geq 3$ and D is a self-orthogonal ternary code of the same length. Thus $L_{C\times D}$ is an even lattice. Since C is τ-invariant, τ induces an isometry of the lattice $L_{C\times D}$. In fact, $\tau(L_{(\lambda,\gamma)}) = L_{(\tau(\lambda),\gamma)}$ for $\lambda \in C$ and $\gamma \in D$. Note that τ is fixed-point-free on $L_{C\times D}$.

Let $V_{L_{C\times D}}$ be the lattice VOA associated with $L_{C\times D}$. The isometry τ of $L_{C\times D}$ lifts to an automorphism of $V_{L_{C\times D}}$ of order 3. We denote it by the same symbol τ. Let $(V_{L_{C\times D}})^\tau$ be the orbifold of $V_{L_{C\times D}}$ by τ. Using the argument in Introduction we obtain three families of simple $(V_{L_{C\times D}})^\tau$-modules. Since

$$(L_{C\times D})^\perp / L_{C\times D} = \{L_{(\lambda,\gamma)} + L_{C\times D} \,|\, \lambda \in C, \gamma \in D\} \cong C^\perp/C \times D^\perp/D,$$

$\{V_{L_{(\lambda,\gamma)}+L_{C\times D}} \,|\, \lambda \in C^\perp/C, \gamma \in D^\perp/D\}$ is a complete set of representatives of isomorphism classes of simple $V_{L_{C\times D}}$-modules by [1]. Now,

$$V_{L_{(\lambda,\gamma)}+L_{C\times D}} \circ \tau = V_{L_{(\tau^{-1}(\lambda),\gamma)}+L_{C\times D}}.$$

Thus $V_{L_{(\lambda,\gamma)}+L_{C\times D}}$ is τ-stable if and only if $\lambda = 0$. Hence the number of τ-stable simple $V_{L_{C\times D}}$-modules is $|D^\perp/D|$. It follows from [12] that all simple τ^i-twisted $V_{L_{C\times D}}$-modules can be obtained by the construction in [3]. Actually, we can describe the simple τ^i-twisted $V_{L_{C\times D}}$-modules $V_{L_{C\times D}}^{T,\eta}(\tau^i)$, $\eta \in D^\perp$ explicitly. It turns out that $V_{L_{C\times D}}^{T,\eta}(\tau^i) \cong V_{L_{C\times D}}^{T,\eta'}(\tau^i)$ if and only if $\eta \equiv \eta' \pmod{D}$.

In this way we have the following simple $(V_{L_{C\times D}})^\tau$-modules.

(1) $V_{L_{(0,\gamma)}+L_{C\times D}}(\varepsilon)$, $\gamma \in D^\perp/D$, $\varepsilon = 0, 1, 2$,

(2) $V_{L_{(\lambda,\gamma)}+L_{C\times D}}$, $0 \neq \lambda \in C/\tau$, $\gamma \in D^\perp/D$,

(3) $V_{L_{C\times D}}^{T,\eta}(\varepsilon)$, $\eta \in D^\perp/D$, $\varepsilon = 0, 1, 2$, $i = 1, 2$,

where C/τ denotes the set of τ-orbits in C.

Toward the classification of simple $(V_{L_{C\times D}})^\tau$-modules, we should try to show that any simple $(V_{L_{C\times D}})^\tau$-module must be isomorphic to one of the above listed known simple $(V_{L_{C\times D}})^\tau$-modules.

Let M be a simple $(V_{L_{C\times D}})^\tau$-module. Since $(V_L^\tau)^{\otimes \ell} \subset (V_{L_{C\times D}})^\tau$, we can study M as a $(V_L^\tau)^{\otimes \ell}$-module. Since V_L^τ is rational, $(V_L^\tau)^{\otimes \ell}$ is also rational. Thus M can be decomposed into a direct sum of simple $(V_L^\tau)^{\otimes \ell}$-modules. Each simple $(V_L^\tau)^{\otimes \ell}$-module is isomorphic to a tensor product of ℓ simple V_L^τ-modules. Let $M^1 \otimes \cdots \otimes M^\ell$ be a simple $(V_L^\tau)^{\otimes \ell}$-module which appear in M as a direct summand. We divide the 30 simple V_L^τ-modules into the following three families.

$$\mathcal{S} = \{V_{L^{(0,j)}}(\varepsilon),\ V_{L^{(c,j)}} \mid j, \varepsilon = 0,1,2\},$$
$$\mathcal{T}_1 = \{V_L^{T,j}(\tau)(\varepsilon) \mid j, \varepsilon = 0,1,2\},$$
$$\mathcal{T}_2 = \{V_L^{T,j}(\tau^2)(\varepsilon) \mid j, \varepsilon = 0,1,2\}.$$

Theorem 4.1. *For a simple $(V_L^\tau)^{\otimes \ell}$-submodule $M^1 \otimes \cdots \otimes M^\ell$ of M, one of the following three cases occurs.*

(1) $M^s \in \mathcal{S}$ *for all* $1 \le s \le \ell$.
(2) $M^s \in \mathcal{T}_1$ *for all* $1 \le s \le \ell$.
(3) $M^s \in \mathcal{T}_2$ *for all* $1 \le s \le \ell$.

Proof. Suppose $M^r \in \mathcal{S}$ and $M^s \in \mathcal{T}_1 \cup \mathcal{T}_2$ for some $1 \le r, s \le \ell$. We use the following fusion rules for simple V_L^τ-modules (cf. [16]).

$$V_L(\varepsilon_1) \times V_{L^{(0,j)}}(\varepsilon_2) = V_{L^{(0,j)}}(\varepsilon_1 + \varepsilon_2),$$
$$V_L(\varepsilon) \times V_{L^{(c,j)}} = V_{L^{(c,j)}},$$
$$V_L(\varepsilon_1) \times V_L^{T,k}(\tau)(\varepsilon_2) = V_L^{T,k}(\tau)(\varepsilon_1 + \varepsilon_2),$$
$$V_L(\varepsilon_1) \times V_L^{T,k}(\tau^2)(\varepsilon_2) = V_L^{T,k}(\tau^2)(2\varepsilon_1 + \varepsilon_2).$$

Let

$$U = V_L(0) \otimes \cdots \otimes \underbrace{V_L(\varepsilon)}^{r} \otimes \cdots \otimes \underbrace{V_L(2\varepsilon)}^{s} \otimes \cdots \otimes V_L(0) \subset (V_{L_{C\times D}})^\tau$$

for $\varepsilon = 1, 2$. That is, the r-th component and the s-th component of U are $V_L(\varepsilon)$ and $V_L(2\varepsilon)$, respectively, and the other components are $V_L(0)$. Since $M^r \in \mathcal{S}$, the difference between the minimal weight of $V_L(\varepsilon) \times M^r$ and that of M^r is an integer. On the other hand the difference between the minimal weight of $V_L(\varepsilon) \times M^s$ and that of M^s belongs to $\pm\frac{1}{3} + \mathbb{Z}$, since $M^s \in \mathcal{T}_1 \cup \mathcal{T}_2$. Then the difference between the minimal weight of $U \times (M^1 \otimes \cdots \otimes M^\ell)$ and that of $M^1 \otimes \cdots \otimes M^\ell$ is not an integer. Since $M^1 \otimes \cdots \otimes M^\ell$ is contained in a simple $(V_{L_{C\times D}})^\tau$-module, this is a contradiction.

Next, suppose $M^r \in \mathcal{T}_1$ and $M^s \in \mathcal{T}_2$ for some $1 \le r, s \le \ell$. Then $M^r \cong V_L^{T,i}(\tau)(\eta_1)$ and $M^s \cong V_L^{T,j}(\tau^2)(\eta_2)$ for some $i, j, \eta_1, \eta_2 \in \{0,1,2\}$. Since $\ell \ge 3$, the above result implies that there is some $t \ne r, s$ such that $M^t \in \mathcal{T}_1 \cup \mathcal{T}_2$. Then $M^t \cong V_L^{T,k}(\tau)(\eta_3)$ or $M^t \cong V_L^{T,k}(\tau^2)(\eta_3)$ for some $k, \eta_3 \in \{0,1,2\}$. Recall the fusion rules $V_L(1) \times V_L^{T,k}(\tau)(\varepsilon) = V_L^{T,k}(\tau)(1+\varepsilon)$ and $V_L(1) \times V_L^{T,k}(\tau^2)(\varepsilon) = V_L^{T,k}(\tau^2)(2+\varepsilon)$. Note that

$$(\text{minimal weight of } V_L^{T,k}(\tau)(1+\varepsilon)) - (\text{ minimal weight of } V_L^{T,k}(\tau)(\varepsilon)) \in \frac{2}{3} + \mathbb{Z},$$
$$(\text{minimal weight of } V_L^{T,k}(\tau^2)(2+\varepsilon)) - (\text{minimal weight of } V_L^{T,k}(\tau^2)(\varepsilon)) \in \frac{1}{3} + \mathbb{Z}.$$

Now, consider

$$U = V_L(0) \otimes \cdots \otimes \underbrace{V_L(1)}^{r} \otimes \cdots \otimes \underbrace{V_L(1)}^{s} \otimes \cdots \otimes \underbrace{V_L(1)}^{t} \otimes \cdots \otimes V_L(0) \subset (V_{L_{C\times D}})^\tau.$$

We see that the difference between the minimal weight of $U \times (M^1 \otimes \cdots \otimes M^\ell)$ and that of $M^1 \otimes \cdots \otimes M^\ell$ belongs to $\pm\frac{1}{3} + \mathbb{Z}$, which is a contradiction. This completes the proof. $\square$

Lemma 4.2. *Assume that $M^s \in \mathcal{S}$ for all $1 \le s \le \ell$ and let $M^s \cong V_{L(0,\delta_s)}(\varepsilon_s)$, $\delta_s, \varepsilon_s \in \{0,1,2\}$ or $M^s \cong V_{L(\mu_s,\delta_s)}$, $\mu_s \in \{a,b,c\}$, $\delta_s \in \{0,1,2\}$. Then $\delta = (\delta_1, \ldots, \delta_\ell) \in GF(3)^\ell$ is orthogonal to D, that is $\delta \in D^\perp$.*

Proof. We note that

$$2 \times (\text{the minimal weight of } V_{L(0,j)}(\varepsilon)) = \begin{cases} 0 \pmod{\mathbb{Z}} & \text{if } j = 0, \\ \frac{1}{3} \pmod{\mathbb{Z}} & \text{if } j = 1,2, \end{cases}$$

$$2 \times (\text{the minimal weight of } V_{L(c,j)}) = \begin{cases} 0 \pmod{\mathbb{Z}} & \text{if } j = 0, \\ \frac{1}{3} \pmod{\mathbb{Z}} & \text{if } j = 1,2. \end{cases}$$

Then

$$2 \times (\text{the minimal weight of } M^1 \otimes \cdots \otimes M^\ell) \equiv \frac{1}{3}\,\mathrm{wt}(\delta) \pmod{\mathbb{Z}}.$$

Take $\gamma = (\gamma_1, \ldots, \gamma_\ell) \in D$ and let

$$U = V_{L(0,\gamma_1)}(0) \otimes \cdots \otimes V_{L(0,\gamma_\ell)}(0) \subset (V_{L_{C\times D}})^\tau.$$

Then by the fusion rules

$$V_{L(0,\gamma_s)}(0) \times V_{L(0,\delta_s)}(\varepsilon_s) = V_{L(0,\gamma_s+\delta_s)}(\varepsilon_s),$$
$$V_{L(0,\gamma_s)}(0) \times V_{L(\mu_s,\delta_s)} = V_{L(\mu_s,\gamma_s+\delta_s)},$$

it follows that

$$2 \times (\text{the minimal weight of } U \times (M^1 \otimes \cdots \otimes M^\ell)) \equiv \frac{1}{3}\,\mathrm{wt}(\gamma+\delta) \pmod{\mathbb{Z}}.$$

Thus $\frac{1}{3}\mathrm{wt}(\gamma+\delta) \equiv \frac{1}{3}\mathrm{wt}(\delta) \pmod{\mathbb{Z}}$. Hence $\mathrm{wt}(\gamma+\delta) \equiv \mathrm{wt}(\delta) \pmod{3\mathbb{Z}}$. Note that $\nu\cdot\nu = \mathrm{wt}(\nu) + 3\mathbb{Z}$ for any $\nu \in GF(3)^\ell$. Thus we have $(\gamma+\delta)\cdot(\gamma+\delta) = \delta\cdot\delta$. This implies that $\gamma\cdot\delta = 0$, since $\gamma \in D$ and D is self-orthogonal. Thus the assertion holds. □

The above theorem and lemma tell us rough structure of an arbitrary simple $(V_{L_{C\times D}})^\tau$-module. Much work still remains to be completed for the classification of all simple $(V_{L_{C\times D}})^\tau$-modules.

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